CS-151 Quantum Computer Science: Problem Set 9

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Guidelines: ***The deadline to return this problem set is 11.59pm on Monday, April 29***. *Remember that you can collaborate with each other in the preliminary stages of your progress, but each of you must write their solutions independently. Submission of the problem set should be via Gradescope only.*

Problem 1. *Recall Shor's 9-qubit code from lecture, where logical qubits are encoded as*

$$|0_L\rangle = \frac{|000\rangle + |111\rangle}{\sqrt{2}} \otimes \frac{|000\rangle + |111\rangle}{\sqrt{2}} \otimes \frac{|000\rangle + |111\rangle}{\sqrt{2}}$$

$$|1_L\rangle = \frac{|000\rangle - |111\rangle}{\sqrt{2}} \otimes \frac{|000\rangle - |111\rangle}{\sqrt{2}} \otimes \frac{|000\rangle - |111\rangle}{\sqrt{2}}$$

a) *Show the procedure for correcting a Y error on the first qubit (a Y_1 error).*

b) *Repeat part (a) for a H error on the second qubit (H_2 error). (hint: you can write H as $H = \frac{X+Z}{\sqrt{2}}$, and recall that measurement collapses the quantum state)*

c) *Analyze the case when there is an error on the first and third qubits ($X_1 X_3$ error). Can Shor's 9-qubit code correct this error?*

Problem 2. *In this problem, we will verify various properties of the five-qubit error-correcting code. As we discussed, to distinguish the 15 possible single-qubit Pauli errors from each other and from the no-error case, we need four distinct syndrome measurements corresponding to Hermitian operators,*

$$M_1 = IZXXZ,$$
$$M_2 = ZIZXX,$$
$$M_3 = XZIZX,$$
$$M_4 = XXZIZ.$$

*Each of the M_i for $i \in \{1, 2, 3, 4\}$ is Hermitian and acts on the i-th qubit. M_i has eigenvalues ± 1 and fulfills $M_i^2 = I$. (**Note: The notation** XYZ **in this problem represents tensor products of Pauli operators, unless mentioned otherwise.**)*

a) *Verify that the above operators M_1, M_2, M_3, M_4 commute with each other.*

b) *We could have defined $M_5 = ZXXZI$ as the fifth syndrome measurement. Verify that M_5 can be generated from the other operators by $M_5 = M_1 \cdot M_2 \cdot M_3 \cdot M_4$ (where $\cdot$ represents matrix products, **not** tensor products) and explain why the outcome of measurement for M_5 can be deduced from the other four. (Hint: Part (a) is relevant to your explanation.)*

c) *View* $S = \langle M_1, M_2, M_3, M_4 \rangle$ *as a stabilizer group. Compute the dimension of the stabilized subspace. Explain why* $\Pi = \frac{(I+M_1)}{2}\frac{(I+M_2)}{2}\frac{(I+M_3)}{2}\frac{(I+M_4)}{2}$ *is a projector. What is the subspace* Π *projects onto?*

d) *Define*

$$|\bar{0}\rangle = \frac{1}{4}(I+M_1)(I+M_2)(I+M_3)(I+M_4)\,|00000\rangle$$
$$|\bar{1}\rangle = \frac{1}{4}(I+M_1)(I+M_2)(I+M_3)(I+M_4)\,|11111\rangle$$

Show that $|\bar{0}\rangle$ *and* $|\bar{1}\rangle$ *are in the code-space stabilized by* $S := \langle M_1, M_2, M_3, M_4 \rangle$.

e) *Show that*

$$\bar{X} = XXXXX$$
$$\bar{Z} = ZZZZZ$$

corresponds to encoded X and Z operations on the codewords $|\bar{0}\rangle$ *and* $|\bar{1}\rangle$*. That is, prove that:*

$$\bar{X}\,|\bar{0}\rangle = |\bar{1}\rangle \qquad \bar{X}\,|\bar{1}\rangle = |\bar{0}\rangle$$
$$\bar{Z}\,|\bar{0}\rangle = |\bar{0}\rangle \qquad \bar{Z}\,|\bar{1}\rangle = -\,|\bar{1}\rangle$$

Problem 3. *In this problem we examine an example of a stabilizer subgroup and the corresponding stabilizer subspace.*

a) *Let* $g_1 = X_1Z_2$, $g_2 = Z_1X_2$ *and let U and W be the linear subspaces of* $\mathbb{C}^4$ *stabilized by* $\langle g_1 \rangle$ *and* $\langle g_2 \rangle$*, respectively. Find U and W.*

b) *Let V be the linear subspace stabilized by* $\langle g_1, g_2 \rangle$*. Find V.*

c) *Show that applying any single qubit Pauli gate to any state in V results in a state which is orthogonal to V. For this problem don't do a brute-force analysis of each case, please use the stabilizer formalism.*

d) *Show that* Y_1Y_2 *also stabilizes V. Again don't do the brute-force calculations, please use the stabilizer formalism.*